

COVERINGS WITH CONNECTED INTERSECTIONS

BY

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If G is a collection of subsets of a set, then a *subintersection* of G is a non-null set which is the common part of the elements of a subcollection of G .

Suppose that a space X is a compact, locally connected, metric continuum. We show that X has a countable basis whose subintersections are connected and uniformly locally connected. In fact, there is a basis for X with the additional property that the collection of closures of elements of this basis is a family of continuous curves such that each subintersection of this family is a continuous curve. This extends a result of Anderson [1]⁽¹⁾ showing that there is a sequence $G_1, G_2, \dots$ such that G_i is a finite $1/i$ -collection of continuous curves covering X and the subintersections of $\sum G_i$ are locally connected.

The notion of partitioning [2, 3, 4] will be used in proving these results. A *partitioning* of X is a finite collection of mutually exclusive connected domains whose sum is dense in X . The partitioning U is a *brick partitioning* if each of its elements is uniformly locally connected and equal to the interior of its closure while the interior of the closure of the sum of two adjacent elements of U is connected and uniformly locally connected. If each element of U is of diameter less than ϵ , U is an ϵ -*partitioning*. In general, if each element of a collection is of diameter less than ϵ , the collection is called an ϵ -*collection*.

The brick partitioning V is a *core refinement* of the brick partitioning U if (a) V is a refinement of U , (b) for each pair of adjacent element u', u'' of U there is a pair of adjacent element v', v'' of V in u' and u'' respectively such that $\bar{v}' + \bar{v}''$ is a subset of the interior of $\bar{u}' + \bar{u}''$, and (c) for each element u of U , the elements of V in u may be ordered $v_0, v_1, \dots, v_n$ such that $\bar{v}_0$ intersects each $\bar{v}_i$ while $\bar{v}_i$ intersects the boundary of u if and only if $i > 0$. We call v_0 a *core element* and $v_1, v_2, \dots, v_n$ *border elements*.

If B is a subset of X and G is a collection of subsets of X , we use $S(B, G)$ to denote the interior of the closure of the sum of the elements of G which have limit points on $\bar{B}$.

We shall use the following result which was proved in [3].

THEOREM 1. *For each brick partitioning U of X and each positive number ϵ , there is a brick ϵ -partitioning V of X which refines U .*

Although the following result is a corollary of Theorem 6, it is given here since its proof is much simpler than that of Theorem 6.

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(¹) Numbers in brackets refer to the references cited at the end of the paper.

THEOREM 2. *The space X has a countable basis whose subintersections are connected and uniformly locally connected.*

Proof. Let $U_1, U_2, \dots$ be a decreasing sequence of brick partitionings of X (U_{i+1} refines U_i and the maximum of the diameters of elements of U_i approaches 0 with $1/i$). The basis is the collection of sets $S(p, U_i)$ for $p \in X$ and $i = 1, 2, \dots$.

If $G = [g]$ is an infinite subcollection of this basis, Πg is either empty or a single point. The conclusion follows in this case. If G is finite, there is a largest integer n such that some $S(p, U_n)$ is an element of G . Any two elements of U_n in Πg are adjacent; moreover, the interior of the closure of their sum is connected, uniformly locally connected, and contained in Πg . Since Πg is the interior of the sum of the closures of adjacent pairs of elements of U_n , it is connected and uniformly locally connected.

THEOREM 3. *For each brick partitioning U of X and each positive number ϵ there is a brick partitioning V of X such that V is a core refinement of U and each border element of V is of diameter less than ϵ .*

Proof. Since U is a brick partitioning of X , there is a positive number ϵ' less than ϵ such that the common boundary of each pair of adjacent elements of U contains a point that is farther than ϵ' from any other element of U . By Theorem 1 there is a brick ϵ' -partitioning U' of X which refines U . For each element u of U , let T_u be a dendron in u which intersects each element of U' in u . There is a brick $(\delta/2)$ -partitioning U'' of X which refines U' where δ is less than the distance between any T_u and the corresponding $X - u$. The core element v_0 of V in u is the set which is maximal with respect to being a connected domain containing T_u and being the interior of the closure of the sum of some elements of U'' whose closures lie in u . The border elements of V in u are components of the intersection of elements of U' with $u - \bar{v}_0$.

THEOREM 4. *For each ordering $u_1, u_2, \dots, u_n$ of the elements of the brick partitioning U of X and each positive number ϵ there is a brick partitioning V of X such that*

- (1) V is a core refinement of U ;
- (2) each border element of V has a diameter of less than ϵ ;
- (3) if v_1, v_2 are adjacent elements of V , one of the sets $S(v_1, U)$, $S(v_2, U)$ contains the other;
- (4) if v_1, v_2 are adjacent elements of V and the element of U containing v_1 precedes in $u_1, u_2, \dots, u_n$ the element of U containing v_2 , then $S(v_1, U)$ contains $S(v_2, U)$.

Proof. We first show that there is a brick partitioning V satisfying (1), (2), and (3). Suppose N is a fixed positive integer. Denote by Lemma N the result obtained by replacing in Theorem 4 conditions (3) and (4) by

(3') if v_1, v_2 are adjacent elements of V , one of the sets $S(v_1, U)$, $S(v_2, U)$ contains the other if each contains N or more elements of U .

Lemma N holds if N is greater than the number of elements in U . We show that it holds for $N = M$ if it holds for $N = M + 1$. Induction then establishes Lemma N; Lemma N for $N = 1$ is Theorem 4 with condition (4) deleted.

Let U' be a brick partitioning of X satisfying the conditions of Lemma N for $N = M + 1$. Define A to be the set of all points p such that $S(p, U)$ contains at least $M + 1$ elements of U . We define a brick partitioning U'' of X whose elements are of two types; (a) each element of U' in $W = S(A, U')$ is an element of U'' ; (b) if u is an element of U , $u - \overline{W}$ is an element of U'' . We note that U'' is a refinement of U and a consolidation of U' . However, it may not be a core refinement of U .

There is a positive number δ_1 so small that if B is a subset of $X - W$ of diameter less than δ_1 , $S(B, U)$ does not contain $M + 1$ elements of U .

Let δ_2 be the minimum of the distances between nonadjacent elements of U' . We note that if B is a subset of X of diameter less than δ_2 and u' is an element of U' with a limit point on B , then $S(u', U)$ contains $S(B, U)$.

We now describe a brick partitioning V which insures that Lemma N holds for $N = M$. Let V' be a brick partitioning of X which is a core refinement of U'' and such that each border element of V' is of diameter less than $\min(\delta_1/2, \delta_2)$. If u'' is an element of U'' of type (b) in an element u of U , the core element of V in u is the interior of the closure of the sum of the elements of V' in u'' whose closures lie in u . The other elements of V are the elements of U' in W and the elements of V' which are not in W and whose closures do not lie in any element of U . We find that V is a core refinement of U .

We now show that the elements of V satisfy conditions (3'). Suppose v_1 and v_2 are two adjacent elements of V such that each of $S(v_1, U)$ and $S(v_2, U)$ contains M or more elements of U . We may suppose that neither v_1 nor v_2 is a core element, for if v_1 is a core element, $S(v_2, U)$ contains $S(v_1, U)$. Hence, if v_i is not in W , it may be supposed to be of diameter less than either $\delta_1/2$ or δ_2 .

If both v_1 and v_2 are subsets of W , they are elements of U' and condition (3') holds for them because each of the sets $S(v_1, U)$, $S(v_2, U)$ contains $M + 1$ elements of U .

If v_1 is a subset of W and v_2 is not, then v_1 is an element of U' . Since the diameter of $\bar{v}_2$ is less than δ_2 , $S(v_1, U)$ contains $S(v_2, U)$.

If neither v_1 nor v_2 is a subset of W , $v_1 + v_2$ is of diameter less than δ_1 . Then $S(v_1 + v_2, U)$ does not contain $M + 1$ elements of U . Hence $S(v_1, U) = S(v_2, U)$.

By induction we find that Lemma N holds for all values N . Since it holds for $N = 1$, there is a sequence $U = V_0, V_1, \dots, V_n$ of brick partitionings of X such that V_{i+1} satisfies (1) and (3) where U is V_i and the diameters of the border elements of V_{i+1} are less than the distance between any nonadjacent elements of V_i and less than the ϵ mentioned in the statement of Theorem 4.

Consider the core partitioning V of U where the core element of V in u_i

is the interior of the closure of the sum of the elements of V_i whose closures lie in u_i . The border elements of V in u_i are the elements of V_i in u_i which are not in this core.

If v_1 and v_2 are two adjacent elements of V in the same element u_i of V , one of the sets $S(v_1, U)$, $S(v_2, U)$ contains the other because if neither v_1 nor v_2 is a core element, then both are elements of V_i . If v_1 and v_2 are adjacent elements, v_1 is in u_i , v_2 is in u_j , and $j > i$, then $S(v_1, U)$ contains $S(v_2, U)$ because the diameter of v_2 is less than the distance between any two nonadjacent elements of V_i . Hence, V satisfies conditions (1), (2), (3), and (4).

THEOREM 5. *Suppose U is a brick partitioning of X and G is a collection of open sets satisfying the following conditions:*

- (a) *each element of G is the interior of the closure of the sum of the elements of a subcollection of U ;*
- (b) *the subintersections of G are connected and uniformly locally connected;*
- (c) *For each subcollection of G , the intersection of the closures of the elements of the subcollection is the closure of the intersection of the elements of the subcollection.*

Then for each positive number ϵ there are a brick partitioning V of X and an ϵ -covering H of X such that $G+H$ satisfies the above conditions (a), (b), and (c) with V substituted for U in condition (a).

Proof. Let U' be a brick $(\epsilon/2)$ -partitioning of X that refines U . We note the G satisfies condition (a) with U' substituted for U .

Since U' has only a finite number of elements, G has only a finite number. Let $u_1, u_2, \dots, u_n$ be an ordering of the elements of U' such that if $i < j$, u_i intersects as many elements of G as u_j does. It follows from condition (c) that if u_i intersects u_j , then each element of G containing u_i also contains u_j .

Let δ be a positive number so small that if B is a subset of X of diameter less than δ , then there exists a point p of X such that $S(p, U')$ contains $S(B, U')$. Suppose V is a core refinement of U' such that V satisfies conditions (3) and (4) of Theorem 4 and the border elements of V are of diameter less than $\delta/2$. Let $v_1, v_2, \dots, v_m$ be an ordering of the elements of V such that v_i precedes v_j provided either (1) v_i lies in an element of U' which precedes the element of U' containing v_j in the ordering $u_1, u_2, \dots, u_n$ or (2) v_i and v_j lie in the same element of U' and $S(v_i, U')$ contains more elements of U' than $S(v_j, U')$ does. We note that if $\bar{v}_i$ intersects $\bar{v}_j$ and $i < j$, then each element of G containing v_i also contains v_j .

For each point p , define $h(p)$ to be the interior of the closure of the sum of all elements of V whose closures lie in $S(p, U')$. Let H be the collection of all such sets $h(p)$. We prove that H is an ϵ -covering of X and that $G+H$ satisfies conditions (a), (b), and (c) with V substituted for U in condition (a).

To prove that H is a covering, consider a point q . Since each border element of V is of diameter less than $\delta/2$, there exists a point p such that $S(p, U')$

contains $S[S(q, V), U']$. Then $h(p)$ contains q . As the elements of U' are of diameter less than $\epsilon/2$, each $S(p, U')$ is of diameter less than ϵ and the elements of H are of diameter less than ϵ .

We next verify condition (b). Let J be a subcollection of $G+H$ and π be the intersection of the elements of J . Since π is the interior of the closure of the sum of the elements of a subcollection of V , it is uniformly locally connected. If J contains no element of H , then π is connected by hypothesis. Suppose J contains an element $h(p)$ of H . Let v_i, v_j be elements of V contained in π and u_r, u_s be the elements of U' containing v_i and v_j respectively. Then $\bar{u}_r \cdot \bar{u}_s$ is not null. Hence π contains v_i, v_j , the core elements of V in u_r and u_s , and the border elements of V whose boundaries intersect the closure of no elements of U' except u_r and u_s . Then π is connected.

Finally, we check condition (c). If $\bar{v}_i$ intersects $\bar{v}_j$ and $i < j$, $S(v_i, U')$ contains $S(v_j, U')$. Hence $\bar{v}_j$ lies in $S(v_i, U')$ and if $h(p)$ contains v_i , it also contains v_j . Hence each element of $G+H$ containing v_i also contains v_j . If p is a point of the intersection of the closures of the elements of J , the last element of $v_1, v_2, \dots, v_m$ having p on its closure is in each element of J . Hence $\bar{\pi}$ contains p and is the intersection of the closures of the elements of J .

THEOREM 6. *The space X has a countable basis G whose subintersections are connected and uniformly locally connected and such that if G' is a subcollection of G , then the intersection of the closures of the elements of G' is the closure of the intersection of the elements of G' .*

Proof. Let G_0 be the covering of X whose only element is X itself, and U_0 be the brick partitioning of X whose only element is X itself. Repeated applications of Theorem 5 give a sequence $G_0, G_1, \dots$ of coverings of X such that G_i ($i \geq 1$) is a $(1/i)$ -covering and such that $G_1 + G_2 + \dots + G_i$ satisfies conditions (b) and (c). Define G to be $\sum G_i$ and the theorem follows. The following result is a consequence of Theorem 6.

THEOREM 7. *For each positive integer i , X is the sum of a finite $(1/i)$ -collection G_i of continuous curves such that each subintersection of $\sum G_i$ is a continuous curve.*

REFERENCES

1. R. D. Anderson, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc. vol. 67 (1949) pp. 451-460.
2. R. H. Bing, *Partitioning a set*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 1101-1110.
3. ———, *Complements of continuous curves*, Fund. Math. vol. 36 (1950) pp. 303-318.
4. E. E. Moise, *Grille decomposition and convexification theorems for compact locally connected continua*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 1111-1121.

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